A bound for the number of different basic solutions generated by the simplex method

(Tomonari Kitahara · Shinji Mizuno)

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The Simplex and Policy-Iteration Methods are Strongly Polynomial for the Markov Decision Problem with Fixed Discount

(Yinyu Ye)

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Introduction

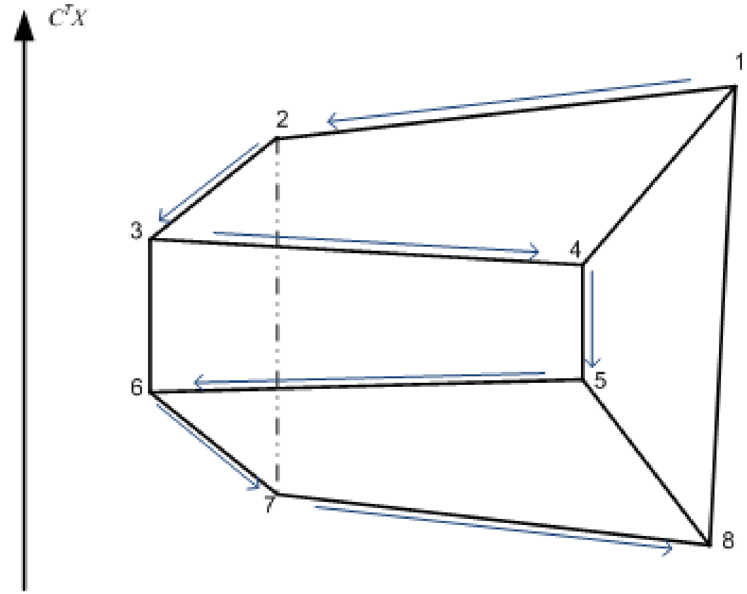
- The simplex method for LP was originally developed by Dantzig.
- In practice, the policy-iteration method, including the simple policy-iteration or Simplex method, has been remarkably successful and shown to be most effective and widely used.
- In spite of the practical efficiency of the simplex method, do not have any good bound for the number of iterations (the bound was only the number of bases $\frac{n!}{m!(n-m)!}$).

Introduction

- Klee and Minty showed that the simplex method needs an exponential number of iterations for an elaborately designed problem LP.
- Melekopoglou and Condon (1990) showed that a simple policy-iteration method, with the smallest index pivoting rule, needs an exponential number of iterations for solving an MDP problem regardless of discount rates.

Introduction

Linier Programming rectangular feasible region



Number of Iterations (or number of vertices generated) is

$$2^n = 8.$$

Result (Tomonari Kitahara · Shinji Mizuno)

The number of different basic feasible solutions (BFSs) generated by the simplex method with Dantzig's rule (the most negative pivoting rule) is bounded by

$$n\left\lceil m\frac{\gamma}{\delta}\log\left(m\frac{\gamma}{\delta}\right)\right\rceil$$

where

- n : The number of variables
- m : The number of constraints
- δ , γ : the minimum and the maximum values of all the positive elements of primal BFSs

When the primal problem is nondegenerate, it becomes a bound for the number of iterations. The bound depends only on the constraints of LP

Result (Tomonari Kitahara · Shinji Mizuno)

If apply the result to an LP where a constraint matrix is totally unimodular and a constant vector $\boldsymbol{b}$ is integral, the number of different solutions generated by the simplex method is at most

$$n\lceil m\|b\|_1 \log(m\|b\|_1)\rceil$$

Result (Yinyu Ye)

The classic simplex method, or the simple policy-iteration method, with the greedy pivoting rule, is a strongly polynomial-time algorithm for MDP with fixed discount rate:

$$\frac{m^2(k-1)}{1-\gamma} . \log\left(\frac{m^2}{1-\gamma}\right)$$

and each iteration uses at most m^2k arithmetic operations, where γ is the fixed discount rate

In general the number of iterations is bounded by

$$\frac{m^2(n-m)}{1-\gamma} . \log\left(\frac{m^2}{1-\gamma}\right)$$

where n is the total number of actions.

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Linier Programming and Its Dual

The standard form of Linier Programming is

$$\min \quad c^T x, \qquad (1)$$

$$\text{subject to} \;\; Ax = b, \quad x \geq 0,$$

where $A \in R^{mxn}$, $b \in R^m$ and $c \in R^n$ are given data, and $x \in R^n$ is a variable vector.

The dual problem of (1) is

$$\max \quad b^T y, \qquad (2)$$

$$\text{subject to} \;\; A^T y + s = c, \quad s \geq 0,$$

where $y \in R^m$ and $s \in R^n$ is a variable vector.

Assumptions

Assume that

- Rank $(A) = m$,
- the primal problem has an optimal solution,
- an initial BFS $\boldsymbol{x}^{\mathbf{0}}$ is available.

Let $\boldsymbol{x}^*$ be an optimal BFS of the primal problem (1), $(\boldsymbol{y}^*; \boldsymbol{s}^*)$ be an optimal solution of the dual problem (2), and $\boldsymbol{z}^*$ be the optimal value of (1) and (2).

Given a set of indices $\boldsymbol{B} \subset \{\mathbf{1}; \mathbf{2}; : : : ; \boldsymbol{n}\}$, we split $\boldsymbol{A}$, $\boldsymbol{c}$, and $\boldsymbol{x}$ according to $\boldsymbol{B}$ and $\boldsymbol{N} = \{\mathbf{1}; \mathbf{2}; : : : ; \boldsymbol{n}\} - \boldsymbol{B}$ like

$$A = [A_B, A_N], \quad c = \begin{bmatrix} c_B \\ c_N \end{bmatrix}, \quad x = \begin{bmatrix} x_B \\ x_N \end{bmatrix},$$

Standard form of LP

The standard form of LP is written as

$$\begin{aligned} \textbf{min} \quad & c_B^T x_B + c_N^T x_N \qquad (3) \\ \textbf{subject to } & A_B x_B + A_N x_N = b, \\ & x_B \geq 0, \qquad x_N \geq 0. \end{aligned}$$

From (1) and (3)

$$Ax = A_B x_B + A_N x_N = b,$$
$$x_B = A_B^{-1} b - A_B^{-1} A_N x_N$$

Then

$$\begin{aligned} c^T x &= c_B^T x_B + c_N^T x_N \\ &= c_B^T (A_B^{-1} b - A_B^{-1} A_N x_N) + c_N^T x_N \\ &= c_B^T A_B^{-1} b + (c_N - A_N^T (A_B^{-1})^T c_B)^T x_N \end{aligned}$$

Basic Feasible Solutions (BFSs)

The primal problem for the basis $\boldsymbol{B} \in \mathrm{B}$ and $\boldsymbol{N} = \{\mathbf{1}; \mathbf{2}; : : : ; \boldsymbol{n}\} - \boldsymbol{B}$ can be written as

$$\begin{aligned} \min \quad & c_{B^t}^T A_{B^t}^{-1} b + (c_{N^t} - A_{N^t}^T (A_{B^t}^{-1})^T c_{B^t})^T x_{N^t}, & (4) \\ \text{subject to} \quad & x_{B^t} = A_{B^t}^{-1} b - A_{B^t}^{-1} A_{N^t} x_{N^t}, \\ & x_{B^t} \geq 0, \qquad x_{N^t} \geq 0. \end{aligned}$$

$\bar{c}_{N^t} = c_{N^t} - A_{N^t}^T (A_{B^t}^{-1})^T c_{B^t}$ be the reduced cost vector, then we can be written as

$$\begin{aligned} \min \quad & c_{B^t}^T A_{B^t}^{-1} b + (\bar{c}_{N^t})^T x_{N^t}, & (5) \\ \text{subject to} \quad & x_{B^t} = A_{B^t}^{-1} b - A_{B^t}^{-1} A_{N^t} x_{N^t}, \\ & x_{B^t} \geq 0, \qquad x_{N^t} \geq 0. \end{aligned}$$

δ = minimum and γ = maximum

Let δ and γ be the minimum and the maximum values of all the positive elements of BFSs. Then for any BFS $\hat{x}$ and any $\boldsymbol{j} \in \{\mathbf{1}; \mathbf{2}; : : : ; \boldsymbol{n}\}$, if $\hat{x}_j \neq 0$, we have

$$\delta \leq \hat{x}_j \leq \gamma, \qquad (6)$$

The values of δ and γ depend only on $\boldsymbol{A}$ and $\boldsymbol{b}$, but not on $\boldsymbol{c}$.

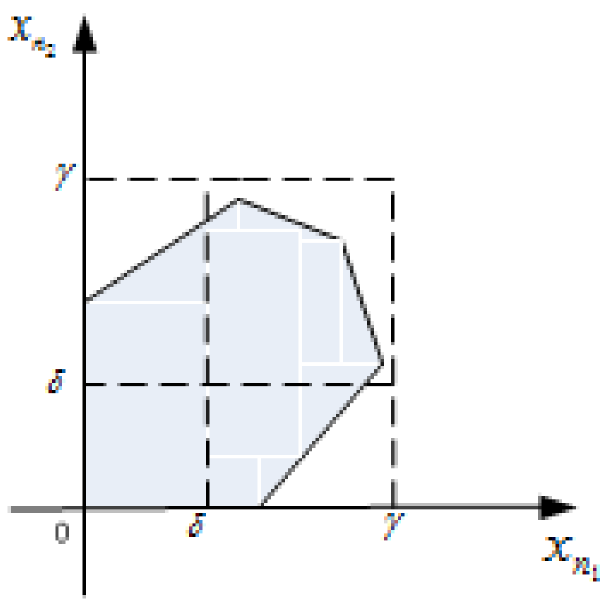


Figure of δ, γ and BFSs (vertices)

Pivoting rule

When $\bar{c}_{N^t} \geq 0$, the current solution is optimal. Otherwise we conduct a pivot. Under the most negative rule, we choose a nonbasic variable whose reduced cost is minimum, i.e., we choose an index

$$\hat{j}^t = \arg\min_{j \in N_t} \bar{c}_{\mathrm{j}}$$

Set $\Delta^t = -\bar{c}_{\hat{j}^t} > 0$, that is, $-\Delta^t$ is the minimum value of the reduced costs

Notations

$\boldsymbol{x}^*$: An optimal basic feasible solution of (1)

$(\boldsymbol{y}^*; \boldsymbol{s}^*)$: An optimal solution of (2)

$\boldsymbol{z}^*$: The optimal value (1) and (2)

$\boldsymbol{x}^t$: The $\boldsymbol{t}$-th iterate of the simplex method

$\boldsymbol{B}^t$: The basis of $\boldsymbol{x}^t$

$\boldsymbol{N}^t$: The nonbasis of $\boldsymbol{x}^t$

$\bar{c}_{N^t}$: The reduced cost vector at $\boldsymbol{t}$-th iteration

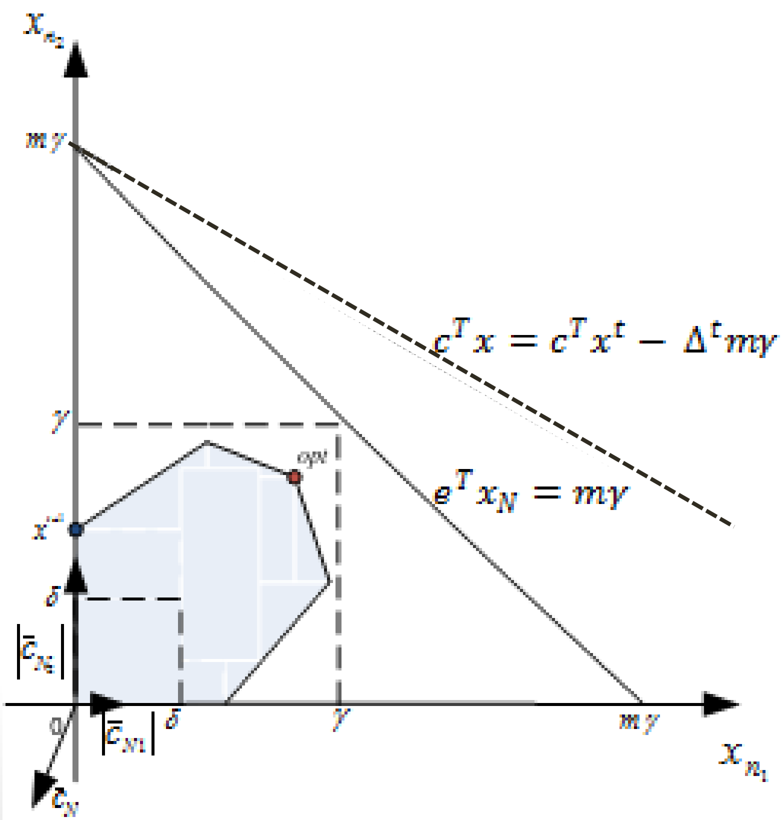
Δ^t : $-\min_{j \in N_t} \bar{c}_{\mathrm{j}}$

$\hat{j}^t$: An index chosen by most negative at t-th iteration

A lower bound of Optimal Value of Simplex method

Lemma 1 *Let z^* be the optimal value of problem* (1) *and* $\mathbf{x}^{\mathbf{t}}$ *be the* **t**-th *iterate generated by the simplex method with the most negative rule. Then we have*

$$z^* \geq c^T x^t - \Delta^t m\gamma. \qquad (7)$$



$$\begin{aligned} \textbf{min} \quad & c_{B^t}^T A_{B^t}^{-1} b + (\bar{c}_{N^t})^T \\ \text{subject to} \quad & x_{B^t} = A_{B^t}^{-1} b - A_{B^t}^{-1} A_{N^t} x_{N^t}, \\ & x_{B^t} \geq 0, \qquad x_{N^t} \geq 0. \end{aligned}$$

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Proof Lemma 1

Proof Let x^* be a basic optimal solution of Problem (1). Then we have

$$\begin{aligned} z^* &= c^T x^* \\ &= c_{B^t}^T A_{B^t}^{-1} b + \bar{c}_{N^t}^T x_{N^t}^* \\ &\geq c^T x^t - \Delta^t e^T x_{N^t}^* \\ &\geq c^T x^t - \Delta^t m\gamma \end{aligned}$$

where the second inequality follows since $\boldsymbol{x}^*$ has at most $\boldsymbol{m}$ positive elements and each element is bounded above by γ.

Reduction Rate (Theorem 1)

A constant reduction of the gap $(\boldsymbol{c}^T\boldsymbol{x}^t - \boldsymbol{z}^*)$ whenban iterate is updated. The reduction rate $\left(1 - \frac{\delta}{m\gamma}\right)$ does not dependent on the objective vector $\boldsymbol{c}$.

Theorem 1 *Let* $\mathbf{x}^\mathbf{t}$ *and* $\mathbf{x}^{\mathbf{t+1}}$ *be the* $\mathbf{t}$-*th and* $\mathbf{(t+1)}$-*th iterates generated by the simplex method with the most negative rule. If* $\mathbf{x}^{\mathbf{t+1}}$, $\mathbf{x}^\mathbf{t}$ *, then we have*

$$\boldsymbol{c}^T\boldsymbol{x}^{t+1} - \boldsymbol{z}^* \leq \left(1 - \frac{\delta}{m\gamma}\right)\left(\boldsymbol{c}^T\boldsymbol{x}^t - \boldsymbol{z}^*\right). \tag{8}$$

Proof Theorem 1

Proof. Let $x^t_{\hat{j}^t}$ be the entering variable chosen at the $\boldsymbol{t}$-th iteration. If $x^{t+1}_{\hat{j}^t} = 0$, then we have $x^{t+1} = x^t$, a contradiction occurs. Thus $x^{t+1}_{\hat{j}^t} \neq 0$, and we have $x^{t+1}_{\hat{j}^t} \geq \delta$ from (6). Then we have

$$
\begin{aligned}
c^T x^t - c^T x^{t+1} &= \Delta^t x^{t+1}_{\hat{j}^t} \\
&\geq \Delta^t \delta \\
c^T x^t - c^T x^{t+1} &\geq \frac{\delta}{m\gamma} \left(c^T x^t - z^*\right) \\
c^T x^{t+1} - z^* &\leq \left(1 - \frac{\delta}{m\gamma}\right) \left(c^T x^t - z^*\right).
\end{aligned}
$$

The best improvement pivoting rule **(Corollary 1)**

The best improvement pivoting rule, the objective function reduces at least as much as that with with the most negative rule. So the next corollary follows

Corollary 1 *Let* $\mathbf{x^t}$ *and* $\mathbf{x^{t+1}}$ *be the* $\mathbf{t}$*-th and* $\mathbf{(t+1)}$*-th iterates generated by the simplex method with the most negative rule. If* $\mathbf{x^{t+1}}$, $\mathbf{x^t}$ *, then also have (8)*

$$\boldsymbol{c}^T\boldsymbol{x}^{t+1} - \boldsymbol{z}^* \leq \left(1 - \frac{\delta}{m\gamma}\right)\left(\boldsymbol{c}^T\boldsymbol{x}^t - \boldsymbol{z}^*\right).$$

Number of solutions **(Corollary 2)**

From Theorem 1 and Corollary 1, we can get an upper bound for the number of different BFSs generated by simplex method.

Corollary 2.

Let $\bar{x}$ be a second optimal BFS of LP (1), that is, a minimal BFS except for optimal BFSs. When we apply the simplex method with the most negative rule (or the best improvement rule) from an initial BFS x^o, is bounded by

$$\left\lceil m\frac{\gamma}{\delta}\log\frac{(c^T x^0 - z^*)}{(c^T \bar{x} - z^*)}\right\rceil \qquad (9)$$

Proof corollary 2

Proof Let x^t be the t-th iterates generated by simplex method and let $\tilde{t}$ be the number of different BFSs appearing up to this iterate. Then we have

$$c^T x^t - z^* \quad \leq \left(1 - \frac{\delta}{m\gamma}\right)^{\tilde{t}} (c^T x^0 - z^*).$$

From (8). If $\tilde{t}$ is bigger than or equal to the number in the corollary, we get

$$c^T x^t - z^* < \left(c^T \bar{x} \quad - z^*\right).$$

Since $\bar{x}$ is a second optimal BFS of LP (1), x^t must be an optimal BFS.

Proof corollary 2

The number of different BFSs generated by the simplex method

$$c^T\bar{x} - z^* \leq \left(1 - \frac{\delta}{m\gamma}\right)^{\bar{t}} (c^T x^0 - z^*)$$

$$1 \leq \left(1 - \frac{\delta}{m\gamma}\right)^{\bar{t}} \left(\frac{c^T x^0 - z^*}{c^T\bar{x} - z^*}\right)$$

$$\log 1 \leq \log\left(1 - \frac{\delta}{m\gamma}\right)^{\bar{t}} + \log\left(\frac{c^T x^0 - z^*}{c^T\bar{x} - z^*}\right)$$

$$0 \leq -\bar{t}\frac{\delta}{m\gamma} + \log\left(\frac{c^T x^0 - z^*}{c^T\bar{x} - z^*}\right)$$

$$\bar{t} \leq \frac{m\gamma}{\delta}\log\left(\frac{c^T x^0 - z^*}{c^T\bar{x} - z^*}\right)$$

An Upper Bound proportional gap (**Lemma 2**)

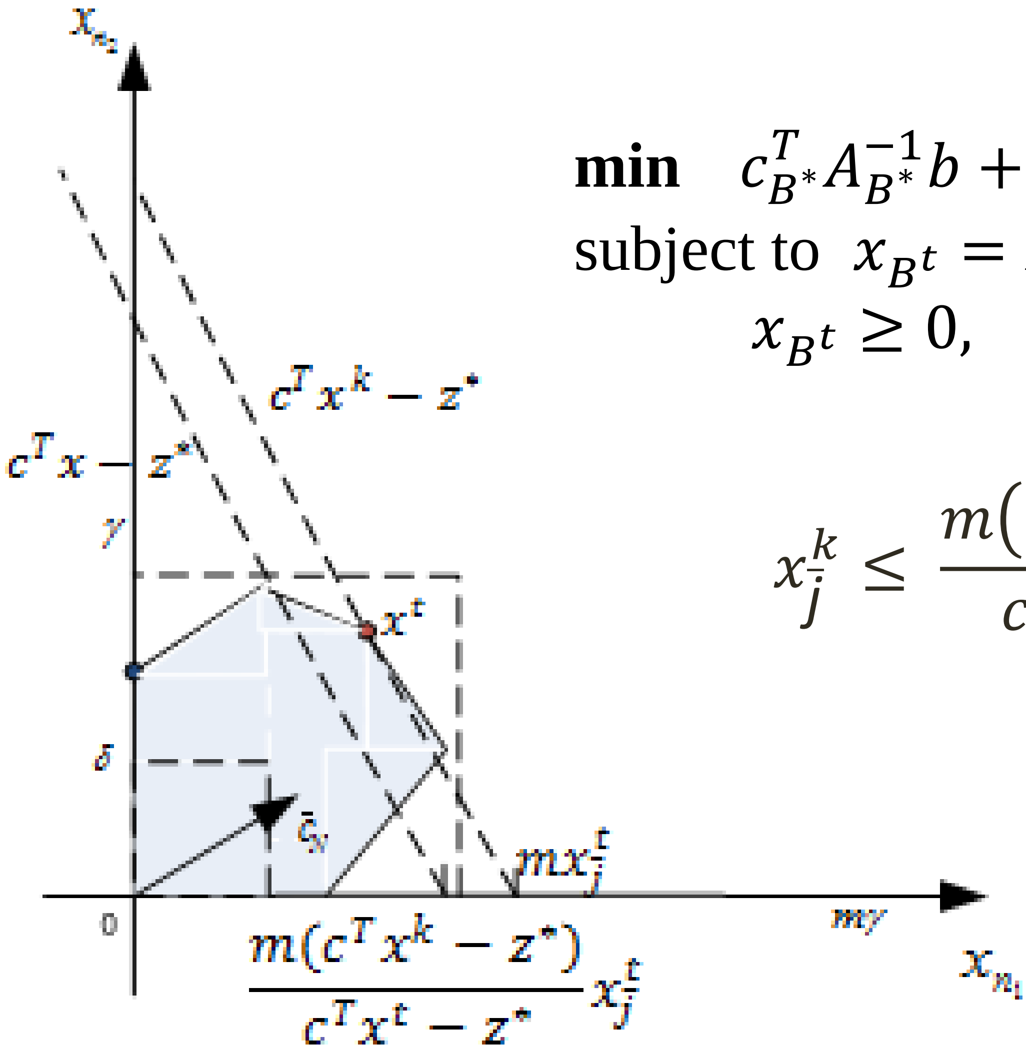
If the current solution is not optimal, there is a basic variable which has an upper bound proportional to the gap between the objective value and the optimal value

Lemma 2 *Let* x^t *be* **t***-th iterate generate by simplex method. If* x^t *is not optimal, there exists* $\bar{j} \in B^t$ *such that* $x_{\bar{j}}^t > 0$ *and*

$$x_{\bar{j}}^k \leq \frac{m(c^T x^k - z^*)}{c^T x^t - z^*} x_{\bar{j}}^t \qquad (10)$$

For any feasible solution **x**.

An Upper Bound proportional gap (**Lemma 2)**



$$\begin{aligned} \textbf{min} \quad & c_{B^*}^T A_{B^*}^{-1} b + (\bar{c}_{N^*})^T \\ \text{subject to} \quad & x_{B^t} = A_{B^*}^{-1} b - A_{B^*}^{-1} A_{N^*} x_{N^*}, \\ & x_{B^t} \geq 0, \qquad x_{N^t} \geq 0. \end{aligned}$$

$$x_j^k \leq \frac{m\left(c^T x^k - z^*\right)}{c^T x^t - z^*} x_j^t$$

Proof Lemma 2

Proof. From primal (1) and dual (2), we have

$$c^T x^t - z^* = c^T x^t - b^T y^*$$
$$= (x^t)^T c - (x^t)^T A^T y^*$$
$$= (x^t)^T (c - A^T y^*)$$
$$= (x^t)^T s^* = \sum_{j \in B^t} x^t s_j^*$$

There exists $\bar{j} \in B^t$ which satisfies

$$c^T x^t - z^* = (x^t)^T s^* \leq m x_{\bar{j}}^t s_{\bar{j}}^*$$

$$m x_{\bar{j}}^t s_{\bar{j}}^* \geq c^T x^t - z^*$$

$$s_{\bar{j}}^* \geq \frac{1}{m x_{\bar{j}}^t} (c^T x^t - z^*)$$

Proof Lemma 2

For any k, the k-th iterate x^k satisfies

$$c^T x^k - z^* = \left(x^k\right)^T s^* = \sum_{j \in B^t} x^k s_j^* \geq x_{\bar{j}}^t s_{\bar{j}}^*$$

which implies

$$x_{\bar{j}}^k \leq \frac{c^T x^k - z^*}{s_{\bar{j}}^*} \leq \frac{m\left(c^T x^k - z^*\right)}{\left(c^T x^t - z^*\right)} x_{\bar{j}}^t$$

Becomes Zero after Iterate (Lemma 3)

Lemma 3 *Let x^t be the **t**-th iterate generated by the simplex method with the most negative rule (the best improvement rule). Assume that x^t is not an optimal solution. Then there exist $\bar{j} \in B^t$ satisfying the following two condition.*

1. $x_{\bar{j}}^t > 0$
2. If the simplex method generates $\left\lceil m\frac{\gamma}{\delta}\log\left(m\frac{\gamma}{\delta}\right)\right\rceil$ different basic solutions after t-th iterate, then $x_{\bar{j}}$ becomes zero and stays zero

Proof Lemma 3

Proof For $k \geq t+1$, let $\widetilde{k}$ be the number of different basic feasible solution appearing between the $(t+1)$-th and k-th iterations. Then from Theorem 1 and Lemma 2, there exist $\bar{j} \in B_t$ which satifies

$$x_{\bar{j}}^{k} \leq m\left(1-\frac{\delta}{m\gamma}\right)^{\tilde{k}} x_{\bar{j}}^{t} \leq m\gamma\left(1-\frac{\delta}{m\gamma}\right)^{\tilde{k}}$$

From definition δ is the minimum value of of all the positive elements of primal BFSs.

$$\delta \leq m\gamma\left(1-\frac{\delta}{m\gamma}\right)^{\tilde{k}}$$

Proof Lemma 3

$$\delta \leq m\gamma\left(1 - \frac{\delta}{m\gamma}\right)^{\tilde{k}}$$

$$1 \leq \frac{m\gamma}{\delta}\left(1 - \frac{\delta}{m\gamma}\right)^{\tilde{k}}$$

$$\log 1 \leq \log\frac{m\gamma}{\delta} + \tilde{k}\ \log\left(1 - \frac{\delta}{m\gamma}\right)$$

$$0 \leq \log\frac{m\gamma}{\delta} - \tilde{k}\ \frac{\delta}{m\gamma}$$

$$\tilde{k}\ \frac{\delta}{m\gamma} \leq \log\frac{m\gamma}{\delta}$$

$$\tilde{k} \leq \frac{m\gamma}{\delta}\log\left(\frac{m\gamma}{\delta}\right) \tag{11}$$

Therefore, if $\tilde{k} > \frac{m\gamma}{\delta}\log\left(\frac{m\gamma}{\delta}\right)$, we have $x_{\bar{j}}^{k} < \delta$, which implies $x_{\bar{j}}^{k} = 0$ from the definition of δ

Bound for the Number of Solutions **(Theorem 2)**

The event described in Lemma 3 can occur at most once for each variable. Thus we get the following result

Theorem 2 *when we apply the simplex method with the most negative rule (the best improvement rule for LP (1) having optimal solutions, we encounter at most*

$$n\left\lceil\frac{m\gamma}{\delta}\log\left(\frac{m\gamma}{\delta}\right)\right\rceil \tag{12}$$

different basic feasible solutions.

Note that the result is valid even if the simplex method fails to find an optimal solution because of a cycling

Primal problem is Nondegenerate **(Corollary 3)**

If the primal problem is nondegenerate, we have $x^{t+1} \neq x^t$ for all t. This observation lead to a bound for the number of iterations of simplex method.

Corollary 3 *If the primal problem is nondegenerate, the simplex method finds an optimal solution in at most*

$$\mathrm{n}\left\lceil \frac{m\gamma}{\delta} \log\left(\frac{m\gamma}{\delta}\right) \right\rceil \text{ iterations}$$

A Totally Unimodular Matrix **(Corollary 4)**

We consider an LP whose constraint matrix A is totally unimodular and all the elements of b are integers, Then all the element of any BFS are interger, so $\delta \geq 1$. Let $(x_B, 0) \in R^m$ x R^{n-m} be a BFS of (1). Then we have $x_B = A_B^{-1}b$. Since A is totally unimodular, all the elements of $A_B^{-1}$are ± 1 or 0. Thus for any $j \in B$ we have $x_j \leq ||b||_1$, which implies that $\gamma \leq ||b||_1$.

Corollary 4 *Assume that the constraint matrix* A *of (1) is totally unimodular and the constraint vector* b *is integral. When we apply the simplex method with The most negative rule fo (1), we encounter at most*

$$n\lceil m||b||_1 \log(m||b||_1)\rceil \qquad (13)$$

different basic feasible solutions.

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Markov Decision Problem (MDP)

The Markov Decision Problem (MDP)

$$\begin{aligned} \min \quad & c_1^T x_1 + c_2^T x_2 \\ \text{subject to} \quad & (I - \theta P_1)x_1 + (I - \theta P_2)x_2 = e, \\ & x_1, x_2 \geq 0 \end{aligned} \tag{14}$$

Where I is the m x m identity matrix, P_1 and P_2 are m x m Markov matrices, θ is a fixed discount rate, and e is the vector of all ones. MDP (14) has the following properties.

1. MDP (14) is nondegenerate.
2. The minimum value of all the positive elements of BFSs is greater than or equal to 1, or equivalently, $\delta \geq 1$.
3. The maximum value of all the positive elements of BFSs is less than or equal to $\frac{m}{1-\theta}$ or equivalently, $\gamma \leq \frac{m}{1-\theta}$.

Number of Iterations for MDP

The result obtain a similar result to **(Yinyu Ye)**

Corollary 5 The simplex method for solving MDP (14) finds an optimal solution in at most

$$\mathrm{n}\left[\frac{m^2}{1-\theta} log\left(\frac{m^2}{1-\theta}\right)\right] \text{ iterations}$$

where $n = 2m$

Result (Yinyu Ye)

$$\frac{m^2(n-m)}{1-\gamma}.\log\left(\frac{m^2}{1-\gamma}\right)$$

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Conclusion

- Constant reduction of the gap:

$$c^T x^{t+1} - z^* \leq \left(1 - \frac{\delta}{m\gamma}\right)(c^T x^t - z^*).$$

- The number of BFSs is bounded by

$$n\left\lceil m\frac{\gamma}{\delta} log\left(m\frac{\gamma}{\delta}\right)\right\rceil$$

- Totally unimodular case: It is bounded by

$$n\lceil m||b||_1 \, log(m||b||_1)\rceil$$

- MDP case: The number of iterations is bounded by

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$$n\left\lceil \frac{m^2}{1-\theta} log\left(\frac{m^2}{1-\theta}\right)\right\rceil$$

  Result (Yinyu Ye)

$$\frac{m^2(n-m)}{1-\gamma} . \log\left(\frac{m^2}{1-\gamma}\right)$$